HOSSAM GHANEM

(30) 4.5 *Summary of Graphical Methods(C)*

Example 9 44 December 21, 2008

Let $f(x) = \dfrac{(x+3)^2}{x}$

(a) Find the intervals on which f is increasing and the intervals on which f is decreasing. Find the local extrema of f, if any.

(b) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward. Find the points of inflection, if any.

(c) Sketch the graph of f.

(10 pts.)

Solution

(a)

$$f(x) = \frac{(x+3)^2}{x}$$

$$f'(x) = \frac{x \cdot 2(x+3) - (x+3)^2(1)}{x^2} = \frac{(x+3)(2x-x-3)}{x^2} = \frac{(x+3)(x-3)}{x^2} = \frac{x^2-9}{x^2}$$

$$f''(x) = \frac{x^2(2x) - (x^2-9)(2x)}{x^4} = \frac{2x(x^2-x^2+9)}{x^4} = \frac{18x}{x^4} = \frac{18}{x^3}$$

(a)

$$f'(x) = \frac{(x+3)(x-3)}{x^2}$$

f ↗ on $(-\infty,-3) \cup (3,\infty)$

f ↘ on $(-3, 3)/\{0\}$

		−3		0		3		
$x-3$	−		−		−		+	3
$x+3$	−		+		+		+	−3
x^2	+		+		+		+	5
	⊕		⊖		⊖		⊕	
	↗		↘		↘		↗	

$f'(x) = 0$

$(x+3)(x-3) = 0$

$x = \pm 3$

$$f(-3) = \frac{(-3+3)^2}{-3} = 0$$

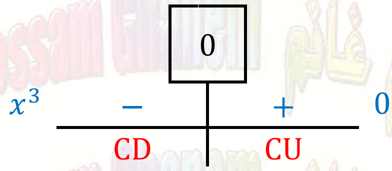
$$f(3) = \frac{(3+3)^2}{3} = \frac{36}{3} = 12$$

at $(-3, 0)$ Maximum local extrema

at $(3, 12)$ Minimum local extrema

(b)

$$f^{\backslash\backslash}(x) = \frac{18}{x^3}$$



The graph of f CD on $(-\infty, 0)$

The graph of f CU on $(0, \infty)$

$$f^{\backslash\backslash}(x) \neq 0$$

No inflection points

(c)

$H.A$:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{(x+3)^2}{x} = \infty$$

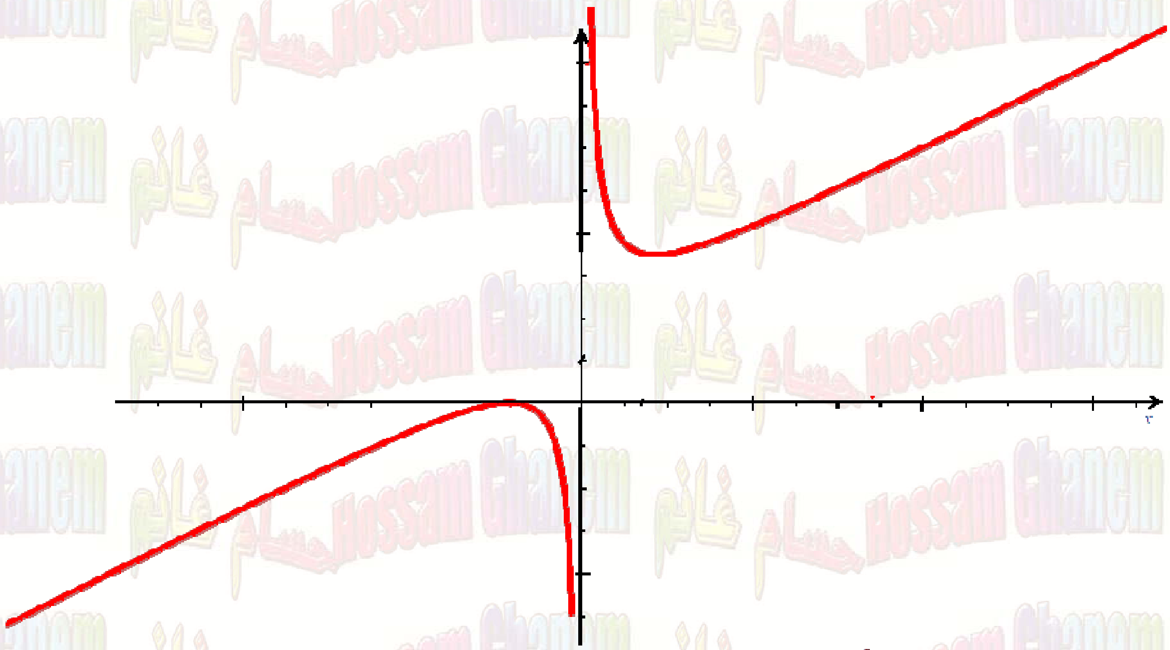
No $H.A$

$V.A$:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{(x+3)^2}{x} = -\infty$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{(x+3)^2}{x} = \infty$$

$V.A$ $x = 0$



Example 10

45 10 May, 2009

Let $f(x) = x^4 - 6x^2 + 2$.

(a) Discuss the symmetry of the graph of $y = f(x)$.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing. Find the local extrema of f, if any.

(c) Find the intervals on which f is concave upwards and the intervals on which f is concave downwards. Find the inflection points of f, if any.

(d) Find the absolute extrema of f on $[-2\,,\ 3\,]$.

(e) Sketch the graph of f. (10 pts)

Solution

(a)

$f(x) = x^4 - 6x^2 + 2$

$f(-x) = (-x^4) - 6(-x^2) + 2 = x^4 - 6x^2 + 2 = f(x)$

$\therefore$ The graph of f *is* symmetric about $y-$axis

(b)

$f^{\backslash}(x) = 4x^3 - 12x = 4x(x^2 - 3)$

$= 4x(x - \sqrt{3})(x + \sqrt{3})$

f ↘ on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

f ↗ on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

		$-\sqrt{3}$		0		$\sqrt{3}$		
x	$-$		$-$		$+$		$+$	0
$x+\sqrt{3}$	$-$		$+$		$+$		$+$	$-\sqrt{3}$
$x-\sqrt{3}$	$-$		$-$		$-$		$+$	$\sqrt{3}$
	⊖		⊕		⊖		⊕	
	↘		↗		↘		↗	

$f(-\sqrt{3}) = (-\sqrt{3})^4 - 6(-\sqrt{3})^2 + 2$

$= 9 - 18 + 2 = -7$

$f(\sqrt{3}) = -7$

$f(0) = 2$

Maximum local extrema at $(0,2)$

Minimum local extrema at $(-\sqrt{3}, -7)$, $(\sqrt{3}, -7)$

(c)

$f^{\backslash\backslash}(x) = 12x^2 - 12 = 12(x^2 - 1) = 12(x - 1)(x + 1)$

		-1		1		
$x-1$	$-$		$-$		$+$	1
$x+1$	$-$		$+$		$+$	-1
	⊕		⊖		⊕	
	CU		CD		CU	

The graph of f CU on $(-\infty, -1) \cup (1, \infty)$

The graph of f CD on $(-1\,, 1\,)$

$f^{\backslash\backslash}(x) = 0$

$12(x - 1)(x + 1) = 0 \quad \rightarrow \quad x = \pm 1$

$f(-1) = (-1)^4 - 6(-1)^2 + 2 = 1 - 6 + 2 = -3$

$f(1) = -3$

inflection point at $(-1, -3)$, $(1, -3)$

(d)

$f(-2) = (-2)^4 - 6(-2)^2 + 2 = 16 - 24 + 2 = -6$

$f(-\sqrt{3}) = -7$

$f(\sqrt{3}) = -7$

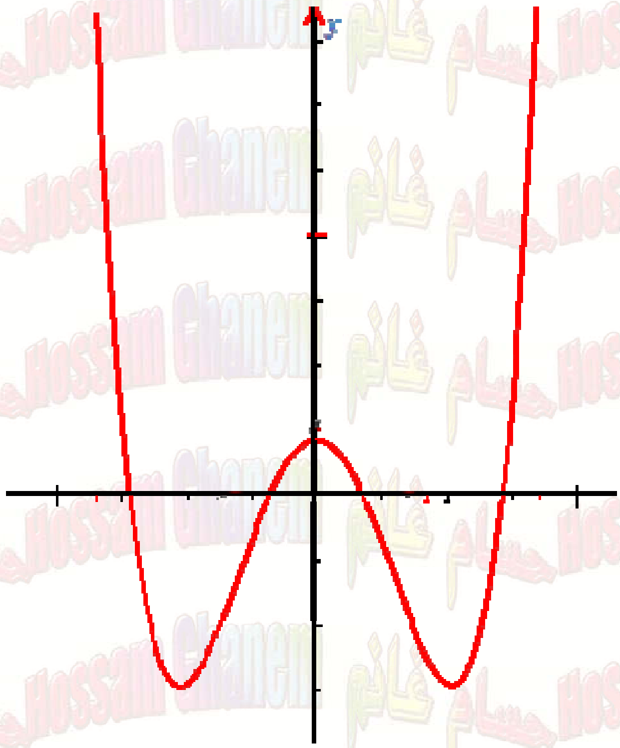
$f(0) = 2$

$f(3) = 3^4 - 6(3)^2 + 2 = 81 - 54 + 2 = 29$

absolute Max. : 29 at (3,29)

absolute Min. : -7 at $(\pm\sqrt{3}, -7)$

(e)



Example 11

46 August 1 , 2009

Let $f(x) = \dfrac{4-x}{(x-2)^2}$ .and given that $f'(x) = \dfrac{x-6}{(x-2)^3}$ and $f''(x) = \dfrac{2(8-x)}{(x-2)^4}$

(a) Find the vertical and horizontal asymptotes for the graph of f, if any.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing. Find the local extrema of f, if any.

(c) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward. Find the points of inflection, if any.

(d) Sketch the graph of f.

(e) Find the maximum and the minimum values of on [3 , 7]. (10 pts)

Solution

(a)

$H.A$:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{4-x}{(x-2)^2} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{4-x}{(x-2)^2} = 0$$

$\therefore y = 0 \quad H.A$

$V.A$:

$$\lim_{x\to2^-} f(x) = \lim_{x\to2^-} \frac{4-x}{(x-2)^2} = \infty$$

$$\lim_{x\to2^+} f(x) = \lim_{x\to2^+} \frac{4-x}{(x-2)^2} = \infty$$

$\therefore \ x = 2 \quad V.A$

(b)

$$f'(x) = \frac{x-6}{(x-2)^3}$$

$f \nearrow$ on $(-\infty, 2) \cup (6, \infty)$

$f \searrow$ on $(2, 6)$

		2		6		
$(x-2)^3$	$-$		$+$		$+$	2
$x-6$	$-$		$-$		$+$	6
	$\oplus$		$\ominus$		$\oplus$	
	$\nearrow$		$\searrow$		$\nearrow$	

$f'(x) = 0$

$x - 6 = 0 \quad \to \quad x = 6$

$$f(6) = \frac{4-6}{(6-2)^2} = -\frac{2}{4^2} = -\frac{2}{16} = -\frac{1}{8}$$

Minimum local extrema at $\left(6, -\dfrac{1}{8}\right)$

(c)

$$f^{\backslash\backslash}(x) = \frac{2(8-x)}{(x-2)^4}$$

The graph of f CU on $(-\infty, 2) \cup (2, 8)$

The graph of f CD on $(8, \infty)$

		2		8		
$(x-2)^4$	+		+		+	2
$8-x$	+		+		−	8
	⊕		⊕		⊖	
	CU		CU		CD	

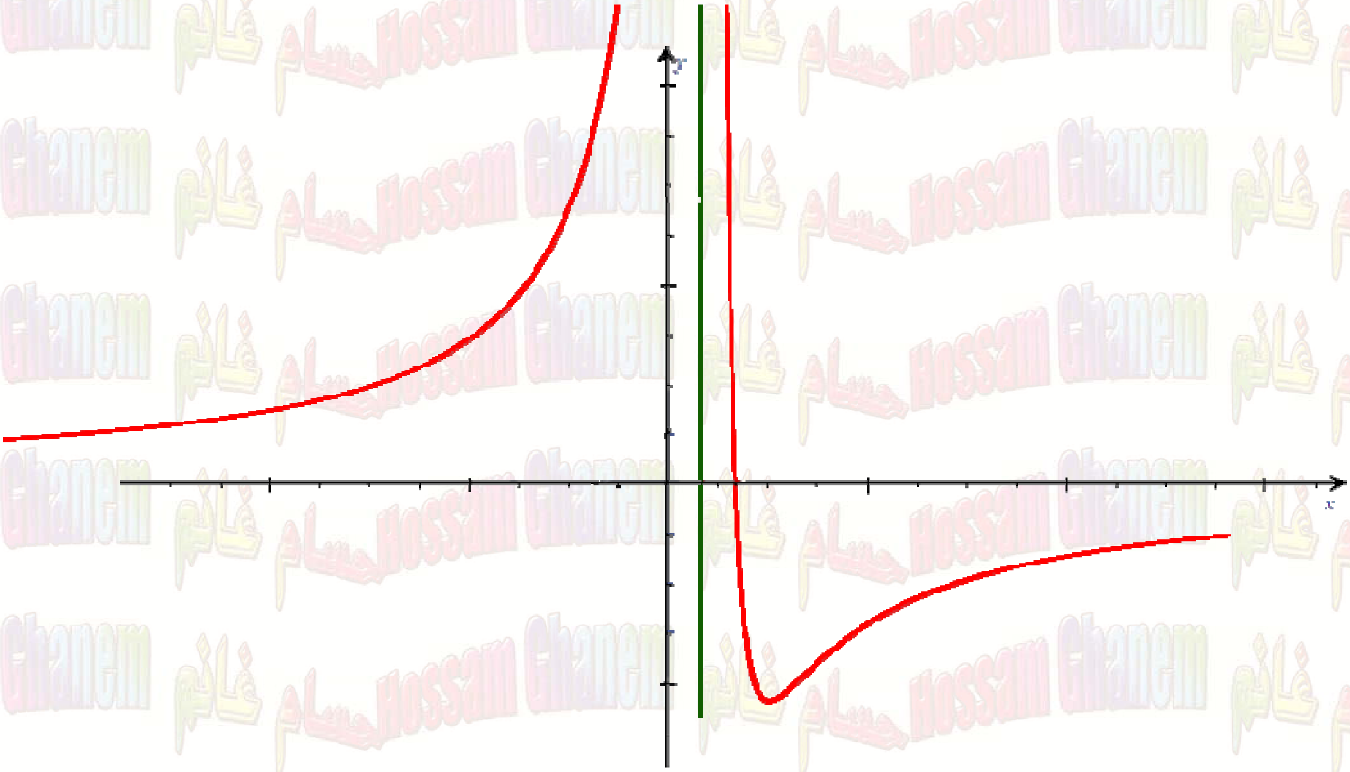
$f^{\backslash\backslash}(x) = 0$

$8 - x = 0 \quad \rightarrow \quad x = 8$

$$f(8) = \frac{4-8}{(8-2)^2} = -\frac{4}{6^2} = -\frac{4}{36} = -\frac{1}{9}$$

$\therefore$ inflction point at $\left(8, -\frac{1}{9}\right)$

(d)



(e)

$$f(3) = \frac{4-3}{(3-2)^2} = 1$$

$$f(6) = \frac{-1}{8}$$

$$f(7) = \frac{4-7}{(7-2)^2} = \frac{-3}{25}$$

absolute Maxi. : 1 at $(3,1)$

absolute Mini. : $\frac{-1}{8}$ at $\left(6, \frac{-1}{8}\right)$

Example 12
47 December 22, 2009

Let

$$f(x)=\frac{4x^2-8x+8}{x^2}$$

(a) Show that $f''(x)=\dfrac{16(3-x)}{x^4}$

(b) Find the vertical and horizontal asymptotes of the graph of f, if any.

(c) Find the intervals on which f is increasing or decreasing and find the local extrema of f, if any.

(d) Find the intervals on which the graph of f is concave up or concave down and find the points of inflection, if any.

(e) Sketch the graph of f. (9 Points)

Solution

(a)

$$f(x)=\frac{4x^2-8x+8}{x^2}=4-\frac{8}{x}+\frac{8}{x^2}=4-8x^{-1}+8x^{-2}$$

$$f'(x)=8x^{-2}-16x^{-3}=\frac{8}{x^2}-\frac{16}{x^3}=\frac{8x-16}{x^3}=\frac{8(x-2)}{x^3}$$

$$f''(x)=-16x^{-3}+16(3)\,x^{-4}=\frac{-16}{x^3}+\frac{16(3)}{x^4}=\frac{-16x+16(3)}{x^4}=\frac{16(-x+3)}{x^4}=\frac{16(3-x)}{x^4}$$

(b)

$H.A$:

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{4x^2-8x+8}{x^2}=4$$

$$\lim_{x\to-\infty} f(x)=4$$

$\therefore\ y=4 \quad H.A$

$V.A$:

$$\lim_{x\to0^-} f(x)=\lim_{x\to0^-}\frac{4x^2-8x+8}{x^2}=\infty$$

$$\lim_{x\to0^+} f(x)=\lim_{x\to0^+}\frac{4x^2-8x+8}{x^2}=\infty$$

$\therefore x=0 \quad V.A$

(c)

$$f'(x)=\frac{8(x-2)}{x^3}$$

f ↗ on $(-\infty,0)\cup(2,\infty)$

f ↘ on $(0,2)$

	$(-\infty,0)$	0	$(0,2)$	2	$(2,\infty)$	
$x-2$	−		−		+	2
x^3	−		+		+	0
	⊕		⊖		⊕	
	↗		↘		↗	

$f'(x)=0$

$x-2=0 \quad \to \quad x=2$

$$f(2)=\frac{4(2)^2-8(2)+8}{(2)^2}=\frac{16-16+8}{4}=2$$

Minimum local extrema at $(2,2)$

(d)

$$f''(x) = \frac{16(3-x)}{x^4}$$

The graph of f CU on $(-\infty, 0) \cup (0,3)$
The graph of f CD on $(3, \infty)$

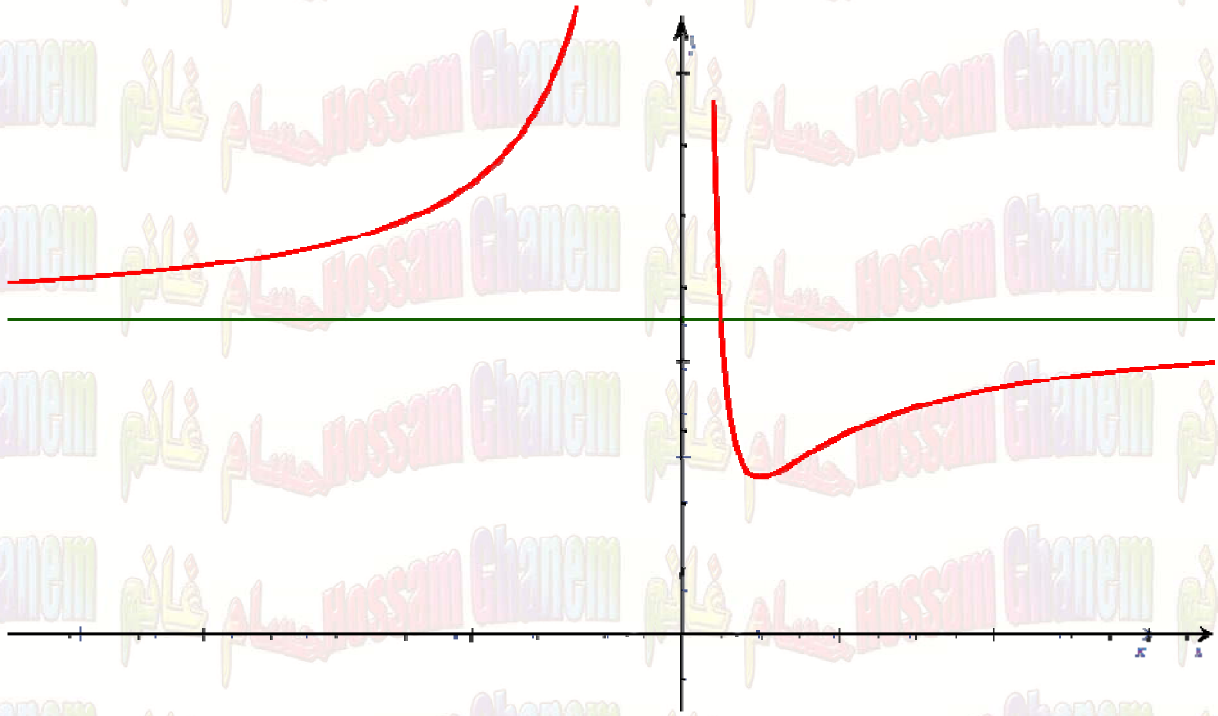
		0		3		
$3-x$	+		+		−	3
x^4	+		+		+	0
	⊕		⊕		⊖	
	CU		CU		CD	

$f''(x) = 0$
$3 - x = 0 \qquad \rightarrow \qquad x = 3$

$$f(3) = \frac{4(3)^2 - 8(3) + 8}{(3)^2} = \frac{36 - 24 + 8}{9} = \frac{20}{9}$$

inflcetion point at $\left(3, \frac{20}{9}\right)$

(e)



Homework

1 30 May 15th, 2003

Let $f(x) = \dfrac{x}{(x+1)^2}$

a) Find the vertical and horizontal asymptotes (if any).

b) Show that $f^{\backslash}(x) = \dfrac{1-x}{(x+1)^3}$. Find the intervals on which f is increasing and the intervals on which f is decreasing and then find the local extrema of f (if any).

c) Given that $f^{\backslash\backslash}(x) = \dfrac{2\,(x-2)}{(x+1)^4}$. Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward. Find the point of inflection (if any).

d) Sketch the graph of f.

2 31 July 31st , 2003

Let $f(x) = \dfrac{3x^2-1}{x^3}$.

a) Find the vertical and horizontal asymptotes of f (if any).

b) Show that $f^{\backslash}(x) = \dfrac{3(1-x^2)}{x^4}$ Find the intervals on which f is increasing and the intervals on which f is decreasing and then find local extrema of f (if any).

c) Given that $f^{\backslash\backslash}(x) = \dfrac{6\,(x^2-2)}{x^5}$. Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward. Find the points of inflection (if any).

d) Discuss the symmetry of the graph of f.

e) sketch the graph of f.